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三状态开关式波动率的美式看跌期权

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摘 要: 本文研究三状态开关式波动率美式看跌期权的定价问题，假设波动率$\sigma(t)$取三个不同的值$\sigma_1, \sigma_2, \sigma_3$，分别对应于股市中的熊市、振荡市和牛市，利用$\Delta$对冲技巧得到了有三条自由边界(即最佳实施边界)的变分不等式模型；作为其应用，本文得到以下结果：若当前股票市场处于熊市(牛市)，则在一定条件下，三状态开关式波动率美式看跌期权的最佳实施边界比标准美式看跌期权的最佳实施边界大(小)，且三状态开关式波动率美式看跌期权的价格比标准美式看跌期权的价格低(高)；若当前股票市场处于振荡市，则在一定条件下，三状态开关式波动率美式看跌期权的最佳实施边界比标准美式看跌期权的最佳实施边界小(大)，且三状态开关式波动率美式看跌期权的价格比标准美式看跌期权的价格高(低)。

关键词: 开关式波动率；变分不等式；自由边界
分类号: **AMS**(2000) 35K85 **中图分类号**: O175.26 **文献标识码**: A

1 引言

近年来，很多学者对开关式波动率期权的相关问题做了大量的工作。Yao等[1]用数值方法计算了有限状态开关式波动率欧式期权的价格。Guo[2,3]导出了二状态开关式波动率欧式期权与二状态开关式波动率永久美式回望期权的稳态解。Guo和Zhang[4]导出了二状态开关式波动率永久美式看跌期权的稳态解，并利用Dynkin公式证明了解的最佳性。Jang和Koo[5]得出了以下结论：二状态开关式波动率美式看跌期权的价格等于二状态开关式波动率欧式看跌期权的价格与合约增加提前实施条款而需要增付的期权金之和，并且证明了此价格是自由边界问题的唯一解；对于二状态开关式波动率永久美式看跌期权而言，它的价格在熊市高于牛市，而最佳实施边界在熊市小于牛市。Buffinton和Elliott[6]给出了有限状态开关式波动率欧式看涨期权价格所满足的偏微分方程(即耦合的Black-Scholes方程)与不光滑的边界条件(即$(S-K)^+$)，并得到了二状态开关式波动率美式看跌期权所满足问题的逼近解。Yao等[7]建立了有限状态开关式波动率欧式期权价格所满足的偏微分方程模型，并给出了光滑的边界条件，在此基础上证明了此问题存在唯一的古典解以及解的收敛性。

在现实的金融市场中，熊市股票波动率通常比牛市大，Yi[8]研究了股票波动率取两个不同值σ_1(熊市)，σ_2(牛市)的情形。实际上，股票市场往往既不处于熊市也不处于牛市，而大约70%左右的时间处于振荡市，本文在上述工作的基础上，讨论了股票波动率取三个不同值σ_1(熊市)，σ_2(振荡市)，σ_3(牛市)的更为一般的情形。

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2 模型的建立

假设股票价格 S_t 遵循几何 Brown 运动

$$\mathrm{d}S_t=\mu S_t\mathrm{d}t+\sigma(\alpha_t)S_t\mathrm{d}W_t,$$

其中 μ 是期望回报率，W_t 是标准布朗运动，σ 是波动率，$\alpha_t=\{1,0,-1\}$，当股票市场处于熊市时 $\alpha_t=1$，处于牛市时 $\alpha_t=-1$，处于振荡市时 $\alpha_t=0$，而且

$$\sigma(\alpha_t)=\begin{cases}\sigma_1, & \alpha_t=1\\ \sigma_2, & \alpha_t=0\\ \sigma_3, & \alpha_t=-1\end{cases}$$

记

$$V_1(S_t,t)=V(S_t,t,\sigma_1),\quad V_2(S_t,t)=V(S_t,t,\sigma_2),\quad V_3(S_t,t)=V(S_t,t,\sigma_3),$$

分别为股票市场处于熊市、振荡市、牛市时美式看跌期权的价格，建立无风险投资组合

$$\pi_{1t}=V_1(S_t,t)-\Delta_{1t}S_t, \tag{1}$$

由于 π_{1t} 是无风险的，故

$$E(\mathrm{d}\pi_{1t})=r\pi_{1t}\mathrm{d}t=r\big[V_1(S_t,t)-\Delta_{1t}S_t\big]\mathrm{d}t, \tag{2}$$

其中 E 为数学期望，r 为无风险利率，若股票市场在时刻 t 处于熊市，假设股票市场在未来短时间内进入牛市的概率为 $\lambda_{13}\mathrm{d}t$，进入振荡市的概率为 $\lambda_{12}\mathrm{d}t$，仍然保持在熊市的概率为 $1-\lambda_{12}\mathrm{d}t-\lambda_{13}\mathrm{d}t$，其中 λ_{12}, λ_{13} 为正常数，则有

$$\begin{aligned}E(\mathrm{d}\pi_{1t}) &= (1-\lambda_{12}\mathrm{d}t-\lambda_{13}\mathrm{d}t)(\mathrm{d}V_1-\Delta_{1t}\mathrm{d}S_t)\\ &\quad+\lambda_{12}\mathrm{d}t\big[(V_2-V_1)-\Delta_{1t}\mathrm{d}S_t\big]+\lambda_{13}\mathrm{d}t\big[(V_3-V_1)-\Delta_{1t}\mathrm{d}S_t\big]\\ &= (1-\lambda_{12}\mathrm{d}t-\lambda_{13}\mathrm{d}t)\Big[\Big(\partial_tV_1+\frac{1}{2}\sigma_1^2S^2\partial_{SS}V_1\Big)\mathrm{d}t+(\partial_SV_1-\Delta_{1t})\mathrm{d}S_t\Big]\\ &\quad+\lambda_{12}\mathrm{d}t\big[(V_2-V_1)-\Delta_{1t}\mathrm{d}S_t\big]+\lambda_{13}\mathrm{d}t\big[(V_3-V_1)-\Delta_{1t}\mathrm{d}S_t\big].\end{aligned}$$

令 $\Delta_{1t}=\partial_SV_1$，忽略 $\mathrm{d}t$ 的高阶无穷小量，可得

$$E(\mathrm{d}\pi_{1t})=\Big(\partial_tV_1+\frac{1}{2}\sigma_1^2S^2\partial_{SS}V_1\Big)\mathrm{d}t+\lambda_{12}\mathrm{d}t(V_2-V_1)+\lambda_{13}\mathrm{d}t(V_3-V_1). \tag{3}$$

合并 (2) 与 (3)，利用最佳性，得

$$\partial_tV_1+\frac{1}{2}\sigma_1^2S^2\partial_{SS}V_1+\lambda_{12}(V_2-V_1)+\lambda_{13}(V_3-V_1)\le r(V_1-S\partial_SV_1),$$

即

$$\partial_tV_1+\frac{1}{2}\sigma_1^2S^2\partial_{SS}V_1+rS\partial_SV_1-rV_1+\lambda_{12}(V_2-V_1)+\lambda_{13}(V_3-V_1)\le 0.$$

令

$$t=T-\tau,\quad \mathcal{L}_iV_i=\partial_\tau V_i-\frac{\sigma_i^2}{2}S^2\partial_{SS}V_i-rS\partial_SV_i+rV_i,\quad i=1,2,3,$$

又由于 $V_1 \geq (K-S)^+$，故有

$$\min\{\mathcal{L}_1V_1 + \lambda_{12}(V_1 - V_2) + \lambda_{13}(V_1 - V_3), V_1 - (K-S)^+\} = 0.$$

用同样的方法得

$$\min\{\mathcal{L}_2V_2 + \lambda_{21}(V_2 - V_1) + \lambda_{23}(V_2 - V_3), V_2 - (K-S)^+\} = 0,$$

与

$$\min\{\mathcal{L}_3V_3 + \lambda_{31}(V_3 - V_1) + \lambda_{32}(V_3 - V_2), V_3 - (K-S)^+\} = 0.$$

在时刻 T (即 $\tau = 0$) 时，与标准美式看跌期权一样，收益函数为 $(K-S)^+$。因此三状态开关式波动率的美式看跌期权问题就是寻求 $V_1(S,\tau)$, $V_2(S,\tau)$, $V_3(S,\tau)$，使其满足

$$\begin{cases} \min\{\mathcal{L}_1V_1 + \lambda_{12}(V_1 - V_2) + \lambda_{13}(V_1 - V_3), V_1 - (K-S)^+\} = 0, & S > 0, \quad 0 < \tau < T, \\ \min\{\mathcal{L}_2V_2 + \lambda_{21}(V_2 - V_1) + \lambda_{23}(V_2 - V_3), V_2 - (K-S)^+\} = 0, & S > 0, \quad 0 < \tau < T, \\ \min\{\mathcal{L}_3V_3 + \lambda_{31}(V_3 - V_1) + \lambda_{32}(V_3 - V_2), V_3 - (K-S)^+\} = 0, & S > 0, \quad 0 < \tau < T, \\ V_1(S,0) = V_2(S,0) = V_3(S,0) = (K-S)^+, \quad S \geq 0. \end{cases} \tag{4}$$

其中

$$\mathcal{L}_iV_i = \partial_\tau V_i - \frac{\sigma_i^2}{2}S^2\partial_{SS}V_i - rS\partial_S V_i + rV_i, \quad i = 1, 2, 3,$$

$\tau = T - t$, t 为时间，T 为期权的到期日，S 为时刻 t 时的股价，$V_i(S,\tau)$ 为在时刻 t 波动率为 σ_i 时的期权价格，r 和 K 分别为利率和敲定价格，σ_1, σ_2, σ_3, λ_{12}, λ_{13}, λ_{21}, λ_{23}, λ_{31}, λ_{32} 均为正常数。根据 Fichera 定理 (参见 [9])，在 $S = 0$ 处不需要给边界条件。

3 $W_{p,loc}^{2,1}$ 解的存在性

为了证明问题 (4) 的解的存在性，建立惩罚函数 $\beta_\epsilon(t)$ (见图 1)，满足下列条件

$$\beta_\epsilon(t) \in C^2(-\infty, +\infty), \quad \beta_\epsilon(t) \leq 0, \quad \beta_\epsilon^{'}(t) \geq 0, \quad \beta_\epsilon^{''}(t) \leq 0,$$

这里 $\beta_\epsilon(0) = -C_0$, ($C_0 > 0$, 定义见 (12))，并且

$$\lim_{\varepsilon \to 0} \beta_\epsilon(t) = \begin{cases} 0, & t > 0, \\ -\infty, & t < 0. \end{cases}$$

由于 $(K-S)^+$ 的光滑性不够好，我们需要定义如图 2 所示的函数 $\pi_\epsilon(t)$。

$$\pi_\epsilon(t) = \begin{cases} t, & t \geq \epsilon, \\ \nearrow, & |t| \leq \epsilon, \\ 0, & t \leq -\epsilon. \end{cases}$$

$$\pi_\epsilon(t) \in C^\infty, \quad 0 \leq \pi_\epsilon^{'}(t) \leq 1, \quad \pi_\epsilon^{''}(t) \geq 0, \quad \lim_{\epsilon \to 0} \pi_\epsilon(t) = t^+.$$

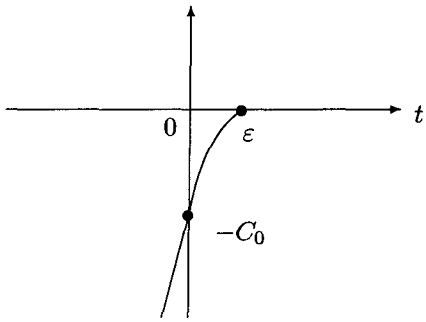

图 1: $\beta_\epsilon(t)$ 的函数图像

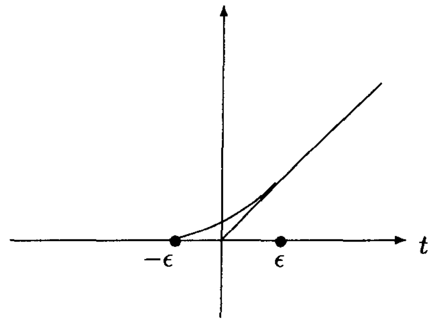

图 2: $\pi_\epsilon(t)$ 的函数图像

首先，我们在区域 $(S_0,+\infty)\times(0,T)$ 上考虑问题 (4)

$$\begin{cases}\min\{\mathcal{L}_1V_1+\lambda_{12}(V_1-V_2)\\ \quad+\lambda_{13}(V_1-V_3),V_1-(K-S)^+\}=0, \quad S>S_0, \quad 0<\tau<T,\\ \min\{\mathcal{L}_2V_2+\lambda_{21}(V_2-V_1)\\ \quad+\lambda_{23}(V_2-V_3),V_2-(K-S)^+\}=0, \quad S>S_0, \quad 0<\tau<T,\\ \min\{\mathcal{L}_3V_3+\lambda_{31}(V_3-V_1)\\ \quad+\lambda_{32}(V_3-V_2),V_3-(K-S)^+\}=0, \quad S>S_0, \quad 0<\tau<T,\\ V_i(S_0,\tau)=K-S_0, \quad i=1,2,3,\\ V_1(S,0)=V_2(S,0)=V_3(S,0)=(K-S)^+, \quad S\geq S_0.\end{cases} \tag{5}$$

其中

$$S_0=\frac{2rK}{2r+\sigma_1^2}$$

是波动率为 σ_1 时永久美式看跌期权的自由边界 (参见 [10])，我们将要证明所有的自由边界都落在区域 $\{S>S_0\}$ 内，因此我们可以通过在区域 $(S_0,+\infty)\times(0,T)$ 上求解来处理算子的退化问题。现在在有界区域 $(S_0,R)\times(0,T)$ 上考虑问题 (5)

$$\begin{cases}\min\{\mathcal{L}_1V_1+\lambda_{12}(V_1-V_2)\\ \quad+\lambda_{13}(V_1-V_3),V_1-(K-S)^+\}=0, \quad S_0<S<R, \quad 0<\tau<T,\\ \min\{\mathcal{L}_2V_2+\lambda_{21}(V_2-V_1)\\ \quad+\lambda_{23}(V_2-V_3),V_2-(K-S)^+\}=0, \quad S_0<S<R, \quad 0<\tau<T,\\ \min\{\mathcal{L}_3V_3+\lambda_{31}(V_3-V_1)\\ \quad+\lambda_{32}(V_3-V_2),V_3-(K-S)^+\}=0, \quad S_0<S<R, \quad 0<\tau<T,\\ V_i(S_0,\tau)=K-S_0,V_i(R,\tau)=0, \quad i=1,2,3,\\ V_1(S,0)=V_2(S,0)=V_3(S,0)=(K-S)^+, \quad S_0\leq S\leq R.\end{cases} \tag{6}$$

其中R为正常数，记$\Omega_T^R=(S_0,R)\times(0,T)$，$\partial_p\Omega_T^R$是区域$\Omega_T^R$的抛物边界，考虑(6)的逼近问题

$$\begin{cases} \mathcal{L}_1V_{1,\epsilon}+\lambda_{12}V_{1,\epsilon}+\lambda_{13}V_{1,\epsilon}+\beta_\epsilon(V_{1,\epsilon}-\pi_\epsilon(K-S)) \\ =\lambda_{12}V_{2,\epsilon}+\lambda_{13}V_{3,\epsilon}, \qquad S_0<S<R,\quad 0<\tau<T, \\ \mathcal{L}_2V_{2,\epsilon}+\lambda_{21}V_{2,\epsilon}+\lambda_{23}V_{2,\epsilon}+\beta_\epsilon(V_{2,\epsilon}-\pi_\epsilon(K-S)) \\ =\lambda_{21}V_{1,\epsilon}+\lambda_{23}V_{3,\epsilon}, \qquad S_0<S<R,\quad 0<\tau<T, \\ \mathcal{L}_3V_{3,\epsilon}+\lambda_{31}V_{3,\epsilon}+\lambda_{32}V_{3,\epsilon}+\beta_\epsilon(V_{3,\epsilon}-\pi_\epsilon(K-S)) \\ =\lambda_{31}V_{1,\epsilon}+\lambda_{32}V_{2,\epsilon}, \qquad S_0<S<R,\quad 0<\tau<T, \\ V_{i,\epsilon}(S,\tau)=\pi_\epsilon(K-S), \qquad i=1,2,3,\quad (S,\tau)\in\partial_p\Omega_T^R. \end{cases} \tag{7}$$

引理1 (a) 对固定的ε, R, $f_2\in L^\infty(\Omega_T^R)$, $f_3\in L^\infty(\Omega_T^R)$, $0\le f_2$, $f_3\le K+\epsilon-S_0$，非线性问题

$$\begin{cases} \mathcal{L}_1V_{1,\epsilon}+\lambda_{12}V_{1,\epsilon}+\lambda_{13}V_{1,\epsilon}+\beta_\epsilon(V_{1,\epsilon}-\pi_\epsilon(K-S))=\lambda_{12}f_2+\lambda_{13}f_3, \quad S_0<S<R,\quad 0<\tau<T \\ V_{1,\epsilon}(S,\tau)=\pi_\epsilon(K-S), \quad (S,\tau)\in\partial_p\Omega_T^R \end{cases}$$

存在唯一解$V_{1,\epsilon}\in W_p^{2,1}(\Omega_T^R)$, $p<+\infty$而且$0\le V_{1,\epsilon}\le K+\epsilon-S_0$。

(b) 对固定的ε, R, $f_1\in L^\infty(\Omega_T^R)$, $f_3\in L^\infty(\Omega_T^R)$, $0\le f_1$, $f_3\le K+\epsilon-S_0$，非线性问题

$$\begin{cases} \mathcal{L}_2V_{2,\epsilon}+\lambda_{21}V_{2,\epsilon}+\lambda_{23}V_{2,\epsilon}+\beta_\epsilon(V_{2,\epsilon}-\pi_\epsilon(K-S))=\lambda_{21}f_1+\lambda_{23}f_3, \quad S_0<S<R,\quad 0<\tau<T \\ V_{2,\epsilon}(S,\tau)=\pi_\epsilon(K-S), \quad (S,\tau)\in\partial_p\Omega_T^R \end{cases}$$

存在唯一解$V_{2,\epsilon}\in W_p^{2,1}(\Omega_T^R)$, $p<+\infty$而且$0\le V_{2,\epsilon}\le K+\epsilon-S_0$。

(c) 对固定的ε, R, $f_1\in L^\infty(\Omega_T^R)$, $f_2\in L^\infty(\Omega_T^R)$, $0\le f_1$, $f_2\le K+\epsilon-S_0$，非线性问题

$$\begin{cases} \mathcal{L}_3V_{3,\epsilon}+\lambda_{31}V_{3,\epsilon}+\lambda_{32}V_{3,\epsilon}+\beta_\epsilon(V_{3,\epsilon}-\pi_\epsilon(K-S))=\lambda_{31}f_1+\lambda_{32}f_2, \quad S_0<S<R,\quad 0<\tau<T \\ V_{3,\epsilon}(S,\tau)=\pi_\epsilon(K-S), \quad (S,\tau)\in\partial_p\Omega_T^R \end{cases}$$

存在唯一解$V_{3,\epsilon}\in W_p^{2,1}(\Omega_T^R)$, $p<+\infty$而且$0\le V_{3,\epsilon}\le K+\epsilon-S_0$。

引理2 对固定的ε, R，问题(7)存在解

$$(V_{1,\epsilon},V_{2,\epsilon},V_{3,\epsilon})\in W_p^{2,1}(\Omega_T^R)\times W_p^{2,1}(\Omega_T^R)\times W_p^{2,1}(\Omega_T^R), \quad p<+\infty.$$

而且

$$0\le V_{i,\epsilon}\le K+\epsilon-S_0, \quad i=1,2,3. \tag{8}$$

引理3 对于问题(7)的解$(V_{1,\epsilon},V_{2,\epsilon},V_{3,\epsilon})$，下面的一致估计成立

$$\pi_\epsilon(K-S)\le V_{i,\epsilon}\le K+\epsilon-S_0, \quad i=1,2,3, \tag{9}$$

$$-C_0\le\beta_\epsilon(V_{i,\epsilon}-\pi_\epsilon(K-S))\le 0, \quad i=1,2,3, \tag{10}$$

$$\partial_\tau V_{i,\epsilon}\ge 0, \quad i=1,2,3. \tag{11}$$

其中

$$C_0=\max\{(2r+\lambda_{12}+\lambda_{13})(K+1),(2r+\lambda_{21}+\lambda_{23})(K+1),(2r+\lambda_{31}+\lambda_{32})(K+1)\}. \quad (12)$$

引理 4 对固定的 $R>0$，问题 (6) 存在唯一解 (V_1,V_2,V_3) 且 $V_i\in C(\overline{\Omega}_T^R)\times W^{2,1}_{p,loc}(\Omega_T^R)$。

由于问题 (6) 的解是唯一的，为了得到 $\partial_S V_i$ 与 $\partial_{SS}V_i$ 的性质，我们考虑 (6) 的另一逼近

$$\begin{cases}\mathcal{L}_1U_{1,\epsilon}+\lambda_{12}U_{1,\epsilon}+\lambda_{13}U_{1,\epsilon}+\beta_\epsilon(U_{1,\epsilon}+\epsilon-\pi_\epsilon(K-S))\\ =\lambda_{12}U_{2,\epsilon}+\lambda_{13}U_{3,\epsilon}, \qquad S_0<S<R,\quad 0<\tau<T,\\ \mathcal{L}_2U_{2,\epsilon}+\lambda_{21}U_{2,\epsilon}+\lambda_{23}U_{2,\epsilon}+\beta_\epsilon(U_{2,\epsilon}+\epsilon-\pi_\epsilon(K-S))\\ =\lambda_{21}U_{1,\epsilon}+\lambda_{23}U_{3,\epsilon}, \qquad S_0<S<R,\quad 0<\tau<T,\\ \mathcal{L}_3U_{3,\epsilon}+\lambda_{31}U_{3,\epsilon}+\lambda_{32}U_{3,\epsilon}+\beta_\epsilon(U_{3,\epsilon}+\epsilon-\pi_\epsilon(K-S))\\ =\lambda_{31}U_{1,\epsilon}+\lambda_{32}U_{2,\epsilon}, \qquad S_0<S<R,\quad 0<\tau<T,\\ U_{i,\epsilon}(S,\tau)=\pi_\epsilon(K-S), \qquad i=1,2,3,\quad (S,\tau)\in\partial_p\Omega_T^R.\end{cases} \quad (13)$$

引理 5

$$\partial_S U_{i,\epsilon}\le 0,\quad \partial_{SS}U_{i,\epsilon}\ge 0,\quad i=1,2,3, \quad (14)$$

最后记 $\Omega_T=(S_0,+\infty)\times(0,T)$，有以下定理。

定理 1 问题 (5) 存在解 (V_1,V_2,V_3)，$V_i\in C(\overline{\Omega}_T)\cap W^{2,1}_{p,loc}(\Omega_T)$，而且

$$(K-S)^+\le V_i\le K-S_0,\quad i=1,2,3, \quad (15)$$

$$\partial_\tau V_i\ge 0,\quad i=1,2,3, \quad (16)$$

$$\partial_S V_i\le 0,\quad i=1,2,3, \quad (17)$$

$$\partial_{SS}V_i\ge 0,\quad i=1,2,3. \quad (18)$$

4 自由边界的性质

定理 2 假设

$$\sigma_1>\sigma_2>\sigma_3, \quad (19)$$

则 1) 当 $\lambda_{23}\ge\lambda_{13}$，$\lambda_{21}\ge\lambda_{31}$ 时有

$$V_1(S,\tau)\ge V_2(S,\tau)\ge V_3(S,\tau),\quad S>S_0,\quad 0<\tau<T, \quad (20)$$

而且

$$V_3^0(S,\tau)\le V_3(S,\tau)\le V_2(S,\tau)\le V_1(S,\tau)\le V_1^0(S,\tau),\quad S>S_0,\quad 0<\tau<T. \quad (21)$$

2) 当 $\lambda_{23}\ge\lambda_{13}$，$\lambda_{21}\ge\lambda_{31}$ 且 $\lambda_{23}=0$ 时有

$$V_2(S,\tau)\ge V_2^0(S,\tau),\quad S>S_0,\quad 0<\tau<T. \quad (22)$$

当 $\lambda_{23} \geq \lambda_{13}$，$\lambda_{21} \geq \lambda_{31}$ 且 $\lambda_{21} = 0$ 时有

$$V_2(S,\tau) \leq V_2^0(S,\tau), \quad S > S_0, \quad 0 < \tau < T. \tag{23}$$

其中 V_1^0，V_2^0，V_3^0 分别是波动率为 $\sigma_1, \sigma_2, \sigma_3$ 时的标准美式看跌期权的价格。

由 (18) 可得

$$\partial_S V_i(S,\tau) \geq \partial_S V_i(S_0,\tau) = -1, \quad S \geq S_0, \quad i = 1,2,3, \tag{24}$$

即

$$\partial_S (V_i - (K - S)) \geq 0, \quad S \geq S_0, \quad i = 1,2,3. \tag{25}$$

由 (25) 定义自由边界

$$S_1(\tau) = \sup\{S \mid V_1(S,\tau) = (K-S)^+\}, \quad 0 < \tau \leq T,$$

$$S_2(\tau) = \sup\{S \mid V_2(S,\tau) = (K-S)^+\}, \quad 0 < \tau \leq T,$$

$$S_3(\tau) = \sup\{S \mid V_3(S,\tau) = (K-S)^+\}, \quad 0 < \tau \leq T.$$

又由于 $\partial_\tau V_i \geq 0$，故有

$$\partial_\tau (V_i - (K - S)) \geq 0, \quad i = 1,2,3. \tag{26}$$

定理 3 $S_1(\tau)$, $S_2(\tau)$, $S_3(\tau)$ 是连续且单调递减的，$S_i(\tau) \in C^\infty(0,T]$，$i = 1,2,3$，并且

1) 当 $\lambda_{23} \geq \lambda_{13}$，$\lambda_{21} \geq \lambda_{31}$ 时有 (见图 3)

$$S_1^0(\tau) \leq S_1(\tau) \leq S_2(\tau) \leq S_3(\tau) \leq S_3^0(\tau), \quad 0 \leq \tau \leq T, \tag{27}$$

$$S_1(0) = S_2(0) = S_3(0) = K. \tag{28}$$

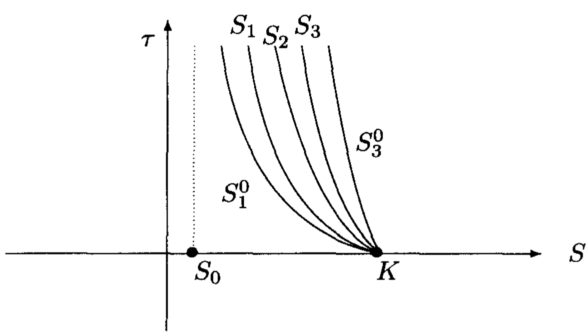

图 3: 最佳实施边界之间的大小关系

2) 当 $\lambda_{23} \geq \lambda_{13}$，$\lambda_{21} \geq \lambda_{31}$ 且 $\lambda_{23} = 0$ 时有

$$S_2(\tau) \leq S_2^0(\tau), \quad 0 \leq \tau \leq T. \tag{29}$$

当 $\lambda_{23} \geq \lambda_{13}$，$\lambda_{21} \geq \lambda_{31}$ 且 $\lambda_{21} = 0$ 时有

$$S_2(\tau) \geq S_2^0(\tau), \quad 0 \leq \tau \leq T. \tag{30}$$

5 唯一性

定义函数类

$$B = \big\{(V_1, V_2, V_3) \in W_{p,loc}^{2,1}(\Omega_T) \times W_{p,loc}^{2,1}(\Omega_T) \times W_{p,loc}^{2,1}(\Omega_T) \mid \lim_{S\to+\infty} V_i = 0,\ \partial_S V_i \text{ 是有界的},\ i = 1,2,3\big\}.$$

附注 1 由 (17) 与 (27) 可得在条件 $\lambda_{23} \geq \lambda_{13}$，$\lambda_{21} \geq \lambda_{31}$ 下有 $-1 \leq \partial_S V_i \leq 0$ 而且

$$\lim_{S\to+\infty} V_i^0(S,\tau) = 0, \quad i = 1,2,3.$$

故由 (21) 在条件 $\lambda_{23} \geq \lambda_{13}$，$\lambda_{21} \geq \lambda_{31}$ 下，

$$\lim_{S\to+\infty} V_i(S,\tau) = 0, \quad i = 1,2,3.$$

故在函数类 B 中讨论唯一性是合理的。

定理 4 问题 (5) 的解在函数类 B 中是唯一的。

附注 2 由定理 1 与定理 4 可得问题 (4) 在区域 $(S_0, +\infty) \times (0, T)$ 上边界条件为

$$V_1(S_0,\tau) = V_2(S_0,\tau) = V_3(S_0,\tau) = K - S_0$$

时存在解，并且在函数类 B 中是唯一的。定义

$$V_1(S,\tau) = V_2(S,\tau) = V_3(S,\tau) = K - S, \quad 0 \leq S < S_0,$$

记

$$\Omega = (0, +\infty) \times (0, T),$$

$$\mathcal{B} = \big\{(V_1, V_2, V_3) \in W_{p,loc}^{2,1}(\Omega) \times W_{p,loc}^{2,1}(\Omega) \times W_{p,loc}^{2,1}(\Omega) \mid \lim_{S\to+\infty} V_i = 0,\ \partial_S V_i \text{ 是有界的},\ i = 1,2,3\big\}.$$

因此，在函数类 $\mathcal{B}$ 中问题 (4) 存在唯一解。

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American Put Options with Three-state Regime Switching Volatility

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Abstract: This paper discusses the fair price of American put options with three-state regime switching volatility. Assuming that the volatility $\sigma(t)$ takes three different values σ_1, σ_2, σ_3 which correspond to bearish, an oscillatory and a bullish stock market, respectively, and applying the Δ hedging technique, we obtain a system of evolutionary variational inequalities which possesses three free boundaries (optimal exercise boundaries). As an application, the following main results is obtained. The optimal exercise boundary of American put options with three-state regime switching volatility in the bearish (bullish) market is higher (smaller) than that of standard American put options and the price of American put options with three-state regime switching volatility in the bearish (bullish) market is smaller (higher) than that of standard American put options on some conditions. The optimal exercise boundary of American put options with three-state regime switching volatility in the oscillatory market is smaller (higher) than that of standard American put options and the price of American put options with three-state regime switching volatility in the oscillatory market is higher (smaller) than that of standard American put options on some conditions.

Keywords: regime switching volatility; variational inequality; free boundary

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